[12.8] Let  be independent 1‑forms in **ℜ***n*. Let  be a p-form and be a q‑form. Show that 

**Proof**: I adopt Juergen Beckmann’s excellent mathematical notation for the antisymmetrization concepts per his solution to this same problem.

**Notation.** Let M = {1, 2, … , *n*}, Pr… u = set of permutations of the *p*-tuple (*r*, … , u) and Pj … m = set of permutations of the *q*-tuple (*j*, … , *m*). Then



Thus



where the brackets inside a bracket [ [*r* … *u*] [*j* … *m*] ] means to antisymmetrize the two antisymmetrizations or, as Juergen states in his proof of this problem, to take the “average” of the two “averages”.

On the other hand,



So the gist of this problem is to show that



that the average of all the terms equals the average of the two sub-averages.

**Lemma.** The sum of the antisymmetrized quantities  equals the sum of the original quantities  That is,



Proof: Fix a *p*-tuple (*r*0, …, *u*0) and consider the RHS term  Where does it appear in the LHS? One place is in the term  where it appears precisely once as . In fact, there are *p*! permutations of and it appears as precisely once in each such permutation, and nowhere else. (Careful examination shows that it always appears as **+**whether ** is even or odd.) Thus the term  appears precisely once on both sides of the equation. This proves that the terms on the RHS are precisely matched by the terms on the LHS, which proves the lemma. ✔

An example is helpful to clarify the notation in (1 – 4) and the lemma (5).

**Example**. Let  be 1-forms in **ℜ**2. Then *p* = 2, *M* = {1, 2 }, so So,





This illustrates both the summation notation and the lemma.

P12 = {1, 2} where



So, for example,



This illustrates the permutation notation and concludes the example.

Continuing the proof, we can use the lemma (5) to rewrite equations (3) and (4):



and



Observe that both expressions (3’) and (4’) have np+q terms that are identical. Thus

 concluding the proof. ✔

Note. Juergen solved this problem by showing directly that antisymmetrization of the *p* and *q* antisymmetrizations equals the single *p*+*q* antisymmetrization. He argued that the inner transpositions are self-cancelling. (Transpositions between one member of { *r*, …, *u }* and one member from { *j* …, *m* } are “**inner transpositions**”. Transpositions that involves terms only from { *r* , …, *u }* or only from { *j* …, *m* }, but not both, are “**outer transpositions**”). This perhaps is what Penrose looking for in this problem.

By collapsing the antisymmetrizations I side-stepped the very difficult issue of showing that the outer antisymmetrizations cancel, leaving just the inner antisymmetrizations.